C 81336 (Pages : 3) Name...

Reg. No....................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, MARCH 2020

(CCSS)

Mathematics

MAT 4E 02—ADVANCED FUNCTIONAL ANALYSIS

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let X be a normed space over k and $A \in BL(X)$. Show that if A is invertible, then

   $\sigma(A^{-1}) = \{k^{-1} : k \in \sigma(A)\}$.

2. Let X be a normed space over K. Show that if

   $x_n \xrightarrow{w} x$ in X and $k_n \to k$ in K, then $k_n x_n \xrightarrow{w} kx$ in X.

3. Define reflexive normed space and show that l^1 is not reflexive.

4. Let X be a normed space and $A \in CL(X)$. Show that every eigen space of A corresponding to a non-zero eigen value of A is finite dimensional.

5. Let (x_n) be a sequence in a Hilbert space H. Show that if $\sum_{n=1}^{\infty} \| x_n \| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges in H.

6. Let H be a Hilbert space and $A \in BL(H)$. Show that $\| A \| = \| A^* \|$.

7. Prove that an orthogonal projection on a Hilbert space H is a positive operator.

8. Let H be a Hilbert space and $A, B \in BL(H)$ with A self-adjoint. Show that $AB = 0$ iff $R(A) \perp R(B)$.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let X be a normed space and $A \in BL(X)$. Show that A is invertible iff A is bounded below and surjective.

10. Show that the dual space of a reflexive normed space is reflexive.

11. Let $<\ ,\ >$ be an inner product on a linear space x and $T : x \to x$ be a linear one-to-one map. Let $<x, y>_T = <T(x), T(y)>$ for $x, y \in X$. Show that $<\ ,\ >_T$ is an inner product on X.

12. Let X be an inner product space. Show that if $E \subset X$ is convex, then there exists at most one best approximation from E to any $x \in X$.

13. Let H be a Hilbert space and $A \in BL(H)$. Show that if A is unitary, then for every orthonormal basis $\{u_\alpha\}$ of H, $\{A(u_\alpha)\}$ and $\{A^*(u_\alpha)\}$ are both orthonormal bases for H.

14. Let H be a finite dimensional Hilbert space over K and $A \in BL(H)$. Show that if there is an orthonormal basis for H consisting of eigen vectors of A, then A is normal.

(4 × 4 = 16 marks)

Part C

Answer **either** *Part (A)* **or** *(B) of each of the following questions.*
Each question carries 12 *marks.*

15. (A) (i) Let x be a normed space and $A \in BL(X)$ be of finite rank. Prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

(ii) Let X be a Banach space over K and $A \in BL(X)$. Show that $\sigma(A)$ is a bounded subset of K.

(B) (i) Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of C_{00} with the norm $\| \|_p$ is linearly isometric to l^q.

(ii) Let X be a finite dimensional normed space. Show that $x_n \xrightarrow{w} x$ in X iff $x_n \to n$ in X.

16. (A) (i) Show that every closed subspace of a reflexive normed space is reflexive.

(ii) State and prove Schwarz inequality.

(B) (i) Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is a compact map iff for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(ii) Let X be a normed space and $A \in CL(X)$. Show that the eigen spectrum and the spectrum of A are countable sets.

17. (A) (i) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Show that $\{u_\alpha\}$ is an orthonormal basis for H iff $x \in H$ and $< x, u_\alpha > = 0$ for all α, then $x = 0$.

(ii) Let $H = L^2([0,1])$. Show that $\{1, \sqrt{2}\cos\pi t, \sqrt{2}\cos 2\pi t, ...\}$ is an orthonormal basis for H.

(B) State and prove Riesz representation theorem.

18. (A) (i) Let H be a Hilbert space and $A \in BL(H)$ be self-adjoint. Show that $\| A \| = \sup\{ |< A(x), x >| : x \in H, \| x \| \le 1\}$.

(ii) Let H be a non-zero Hilbert space and $A \in BL(H)$ be self-adjoint. Show that :

$$\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A].$$

(B) (i) Prove that a Hilbert-Schmidt operator on a Hilbert space H is compact.

(ii) Let H be a Hilbert space and $A \in BL(H)$. Show that A is compact iff A^*A is compact.

(4 × 12 = 48 marks)