C 81339 (Pages : 3) Name.....................................

Reg. No.................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION MARCH 2020

(CCSS)

Mathematics

MAT 4E 08—MEASURE AND INTEGRATION

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let X is a measurable space, Y is topological space, and f is a mapping of X into Y. When do you say that f is measurable ? Given example of a measurable function.
2. Given example of a non-measurable set in a measure space.
3. Show that if $f \in L^1(\mu)$ then f is a finite valued almost every where.
4. Define Borel measure on a space X.
5. Suppose μ, λ, λ_1 and λ_2 on a σ-algebra $\mathcal{M}$ and μ is positive. If $\lambda_1 \perp \mu$ and $\mu_2 \perp \mu$ then prove that $\lambda_1 + \lambda_2 \perp \mu$.
6. Let μ and λ be positive measures on a σ algebra $\mathcal{M}$. Is it true that "Either μ is absolutely continuous with respect to λ or λ is absolutely continuous with respect to μ. Justify your answer.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 marks.

7. Let X = $\{a, b, c, d\}$ and let F $: \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$ be a collection of sub-sets of X. Find the σ algebra generated by $\mathcal{F}$.
8. Let X be a measurable space and f be a complex measurable function on X. Prove that there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha |f|$.

Turn over

9. Let f be a measurable function on $(X, \mathcal{M}, \mu)$. Prove that any function g on X which is equivalent to f almost everywhere then g is measurable.

10. Find $\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{\frac{x}{2}} \, dx$.

11. Prove that Lebesgue measure is regular.

12. Prove that every compact subset of $\mathbb{R}^1$ is the support of a Borel measure.

13. Prove that if μ is a complex measure on X, then $|\mu|(X) < \infty$.

14. Is Hahn decomposition of a measure unique. Justify your answer.

(5 × 4 = 20 marks)

Part C

Answer **either** A **or** B *of each of following questions.*
Each question carries 16 *marks.*

15. (A) (i) Prove that for every real valued non-negative mesurable function $f(x)$ on measureable space can be approximated by non-negative monotonically increasing simple measurable functions $s_n(x)$ on measurable space which converges point wise to $f(x)$.

(4 marks)

(ii) State and prove Lebesgue Monotone Convergence Theorem. (6 marks)

(iii) Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f_n : X \to [0, \infty]$ is measurable, for $n = 1, 2, 3, \ldots$ and $f(x) = \sum_{n=1}^{\infty} f_n(x), x \in X$ then prove that :

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

(6 marks)

Or

(B) (i) State and prove Dominated Convergence theorem. (6 marks)

(ii) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{M}^*$ be the collection of all $E \subset X$ for which there exists set A and B such that $A \subset E \subset B$ and $\mu(B/A) = 0$. Prove that $\mathcal{M}^*$ is a σ algebra containing $\mathcal{M}$. Also prove that there is unique measure μ^* on $\mathcal{M}^*$ such that $\mu^* |_{\mathcal{M}} = \mu$.

(10 marks)

16. (A) (i) State and prove Vitali- Caratheodory theorem. (9 marks)

(ii) Prove that Lebesgue measure is completion of Borel Measure. (7 marks)

Or

(B) (i) State and prove Lusin's theorem. (9 marks)

(ii) Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a linear transformation. Then prove that there exists a non-negative real number Δ (T) such that, for every $E \subset \mathcal{M}$.

(a) $T(E) \in \mathcal{M}$.

(b) $m(T(E)) = (\Delta(T))(m(E))$.

(7 marks)

17. (A) State and prove Lebesgue Radon Nikodym theorem.

Or

(B) Let $1 \leq p < \infty, \mu$ is a σ finite measure on a measurable space X and let ϕ be a bounded linear functional on $L^P(\mu)$. Prove that there is a unique $g \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that :

$$\phi(f) = \int_X fg \, d\mu.$$

[3 × 16 = 48 marks]