Report on

Group Action-An Introduction

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GROUP ACTIONS: AN INTRODUCTION

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Abstract. This note is prepared as part of 15 minutes presentation in the Interdisciplinary Refresher course in Mathematics, Statistics & Computer Science held at Academic Staff College, Kannur University during 8-21, December 2021.
One of my areas of interest is Geometric Group theory. Group actions play an important role in connecting Group theory with Geometry and vice versa. In this note, I will introduce "Group Actions".

1. Introduction

One of the foundational roots of group theory was the quest of solutions of polynomial equations of degree greater than 4. Galois found that if $\alpha_1, ..., \alpha_n$ are the n-roots of a polynomial equation, then there is a set of permutations (bijections) of the roots having a nice algebraic structure (called Group) with certain properties and Galois was the first to use the word "group". Group theory is developed on the basis of symmetries from geometry. A symmetry of a geometric object is a transformation under which the object is invariant. For example, consider a unit square and label its vertices as 1,2,3,4. What are the symmetries of this object? Of course, the trivial transformation, say R_0, is a symmetry since it does nothing to the object. The anti-clockwise rotations R_θ for $\theta = 90, 180, 270$ about the centre of the square (palace the square in the plane so that its centre is the origin of the plane) are also symmetries of this object. The horizontal (H), vertical (V), diagonal (D) and anti-diagonal (D') reflections are also keep the object invariant. A Composition of any two of these symmetries is again a symmetry. It is obvious that composition of a rotation and a reflection and composition of two reflections are rotations and belongs to the eight symmetries listed above. Thus the symmetries of a square are $\{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$ and this set is closed under composition (namely a Group). In general, symmetries of a regular n-gon, called the Dihedral groups, denoted D_n, contains $2n$ elements, n rotations and n reflections. Since a rigid motions of a regular n-gon is determined by rigid motions of its set of vertices, these symmetries are permutations of its vertices $\{1, 2, ..., n\}$. Also the elements of the symmetric group S_n on $\{1, 2, ..., n\}$ are its permutations.
More abstractly, if we are given a set X, the $Sym(X)$ of all permutations of X is a group under composition and if $|X| = n$ then $Sym(X) = S_n$. So the abstract symmetric groups $Sym(X)$ really do arise naturally.

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2. Group Actions

Let $G = \{e, g_1, g_2, ..., g_n\}$ be a group, finite for the time being. Observe the group table. We can see that the row agains a fixed element $g \in G$ is $\{ge, gg_1, gg_2, ..., gg_n\}$. Since G is a group this row contains all the elements of G. Thus for each $g \in G$, there is a permutation (bijection) l_g of G given by $l_g(g_i) = gg_i$. Note that, finiteness of G is not needed to prove l_g is a permutation of G. Thus, summarising, for each $g \in G$, there is a permutation $l_g \in Sym(G)$ such that $l_e = id_X$, $l_g \circ l_h = l_{gh}$ for all $g, h \in G$. This gives us an embedding (an injective homomorphism) of G into $Sym(G)$ via $\pi : G \to Sym(G)$ given by $g \mapsto l_g$. This is the idea of Cayley's theorem".

Theorem 2.1. *Every group is a subgroup of a permutation group.* □

Allowing an abstract group to behave as permutations of a set is a very useful idea, and if it happens we say the group is acting on that set. So we define action of a group on a set as:

Definition 2.2. *An action of a group G on a set X is a choice, for each $g \in G$, a permutation π_g on X such that*

(1) π_e *=identity map*
(2) $\pi_g \circ \pi_h = \pi_{gh}$ *for all* $g, h \in G$

The basic idea in a group action is that the elements of a group are viewed as permutations of a set in such a way that composition of the corresponding permutations matches the group multiplication.

If we denote the effect $\pi_g(x)$ of a $g \in G$ on $x \in X$ by $g.x$, we can rewrite the axioms of the action of G on X as :

(1) $e.x = x$ for all $x \in X$
(2) $g.(h.x) = (gh).x$ for all $x \in X, g, h \in G$.

Another way to see actions of a group on a given set is :

$$\{\text{Actions of } G \text{ on } X\} \leftrightarrow \{\text{homomorphisms } \phi : G \to Sym(X)\}$$

given by for $g \in G, \phi(g) := \pi_g$ for a given action π and if ϕ is a homomorphism, define $\pi_g := \phi(g)$.

2.1. Examples:

(1) Every group G acts on itself trivially : $g.x := x, \forall x \in G$.
(2) Every group acts on itself by left multiplication (which is the idea of Cayley's theorem) given by $g.h := gh$.
(3) The symmetric group S_n acts of the set of all polynomials in n-variables by permuting the variables: for $\sigma \in S_n, f(x_1, x_2, ..., x_n)$ the actions is given by $\sigma.f(x_1, x_2, ..., x_n) := f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$. If we take f to be a homogeneous polynomial of degree 1, say $f = \sum c_i x_i$, then $\sigma.f = \sum c_i x_{\sigma(i)} = \sum c_{\sigma^{-1}(i)} x_i$.

(4) Another tricky example is the action of S_n on $\mathbb{R}^n$ by permuting the coordinates, that is $\sigma.(c_1, c_2, ..., c_n) := (c_{\sigma(1)}, c_{\sigma(2)}, ..., c_{\sigma(n)})$. This is not an action since $(\sigma_1\sigma_2).x \neq \sigma_1.(\sigma_2.x)$, for example, take $n = 3, x = (5, 6, 7)$, $\sigma = (123) = (12)(23) =: \sigma_1\sigma_2 \in S_3$, $(123)(5, 6, 7) = (6, 7, 5)$ but $(12)(23)(5, 6, 7) = (12)(5, 7, 6) = 7, 5, 6)$. From the above example we can see that if we define
$\sigma.(c_1, c_2, ..., c_n) := (c_{\sigma^{-1}(1)}, c_{\sigma^{-1}(2)}, ..., c_{\sigma^{-1}(n)})$
then it is the action of S_n on the homogeneous polynomials since $\sigma.x = \sigma.(c_1, c_2, ..., c_n)$ $= \sigma.(\sum c_ie_i) = \sum c_ie_{\sigma(i)} = \sum c_{\sigma^{-1}(i)}e_i = (c_{\sigma^{-1}(1)}, c_{\sigma^{-1}(2)}, ..., c_{\sigma^{-1}(n)})$ where e_i is the standard basis vector.

3. Application

Question: How many different polynomials can be obtained from a given polynomial $f(x_1, ..., x_n)$ on n-variables after applying all possible permutations of its variables?

That is we are looking for the number of polynomials $\sigma.f, \sigma \in S_n$. To answer this, we need the following observations:
Number of permutations on n-variables is $n!$ and S_n acts on the set of polynomials by permuting the variables.

Definition 3.1. *: Let G acts on a set X, $x \in X$. The orbit of x is the places where the point x can be moved by the action of G, ie., $Orb(x) := \{g.x | g \in G\}$. The stabilizer of x is the set of group elements that fix the point x, ie., $Stab(x) := \{g \in G | g.x = x\}$. It is easy to see that the stabilizer is always a subgroup of G and the set X is partitioned by the distinct orbits and thus $X = \cup Orb(x)$, thus if X is a finite set $|X| = \sum |Orb(x)|$, sum is taken over distinct orbits. This equation is the class equation of the action.*

Observe that there is a surjection from $Orb(x) \to G$ given by $g.x \mapsto g$. This is not an injection since $g.x = g_2.x \implies g_2^{-1}g_1.x = x \implies g_2^{-1}g_1 \in Stab(x)$. Thus, this map induces a bijection $Orb(x) \to G/Stab(x)$ given by $g.x \mapsto gStab(x)$ and hence, if G is a finite group, we can say $|Orb(x)| = |G/Stab(x)|$ or $|Orb(x)| \times |Stab(x)| = |G|$. This is known as the Orbit-Stabilizer theorem.

Now the answer to the question is immediate from the orbit-stabilizer theorem applying to the action of S_n on the set of polynomials on n-variables. The required number is a factor of $n!$ and is equal to $n!/|Stab(f)|$. This was the original version of Legrange's theorem.

Considering various actions of a group G on sets like a subgroup H of G, set of all subgroups of G, the quotient set G/H etc, we can see that familiar terms like a left coset, right coset, normalizer of a subgroup, centralizer of an element/ subgroup, a normal subgroup, conjugacy classes etc. as either stabilizer, orbit or fixed points of the corresponding action.

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