δ-Hyperbolic Spaces

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Geometric Group Theory

- Geometric Group theory is a relatively new area of Mathematics.
- It had an explosive growth in the last thirty years with the path breaking work of M Gromov.
- Now the subject is at the meeting place of geometry, topology and combinatorial group theory.
- This rich area has taken its tools from many branches such as combinatorial group theory, Lie groups, topology and so on.

Geometric Group Theory

We will see some basic notions of geometric group theory here.

- When viewed from a distance, the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ looks like the plane. Here, one is a group and the other is a metric space.
- This is a simple instance introduced by M Gromov, " that viewed from a distance, many objects can be regarded as geometric objects".
- A very important class of such groups is those whose large-scale behaviour (coarse geometry) is similar to that of negatively curved spaces (known as hyperbolic spaces).

Groups in Geometry

We consider geodesic spaces. ie: spaces where every pair of points is connected by a geodesic (a curve every sub-arc of which has length the distance between its endpoints).

- How a group Γ appears in geometry?
- which typically means that that Γ acts on a metric space X of some sort.

  The geometry of the space X reflects some geometry of the group.

Groups in geometry

- $\mathbb{Z}$ looks like $\mathbb{R}$ when viewed from a distance.
- Note that $\mathbb{Z}$ acts on $\mathbb{R}$ with compact quotient $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$.

  In fact the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$ with $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. And $\mathbb{Z}$ and $\mathbb{R}$ are the same in large-scale.

Groups in Geometry

In geometric group theory, groups are studied via their actions on metric spaces via isometries.

Note: Action of a group Γ on X:
$. : \Gamma \times X \to X$ with

- $(gh).x = g.(h.x)$ and
- $1.x = x, \forall x \in X, g, h \in \Gamma$

ie the map $\phi : \Gamma \to Sym(X)$ given by $\gamma \mapsto \{x \mapsto \gamma.x\}$ is a group homomorphism. If $\phi(\Gamma) \subset Isom(X)$, we say the action is by isometries.

We will construct a geodesic metric space out of a group (finitely generated).

Let Γ be a group with a finite generating set $\mathcal{A}$.

We always assume that $1 \notin \mathcal{A}$. Let $\gamma \in \Gamma$, write $\gamma = a_1 a_2 \cdots a_n$ with $a_i \in \mathcal{A} \cup \mathcal{A}^{-1}$. We call *length* of γ, denoted $l_{\mathcal{A}}(\gamma)$, length of the shortest such words representing γ.

Equip Γ with the *word metric* $d_{\mathcal{A}}$ with respect to $\mathcal{A}$ defined by $d_{\mathcal{A}}(\gamma, \gamma') = l_{\mathcal{A}}(\gamma^{-1}\gamma')$ for any two elements $\gamma, \gamma' \in \Gamma$.

This metric is invariant with respect to the action of Γ by left multiplication on itself.

Cayley Graph

The *Cayley graph* $C_{\mathcal{A}}(\Gamma)$ is the graph whose set of vertices is Γ and there is an edge joining γ and γ' if and only if $\gamma' = \gamma a$ for some $a \in \mathcal{A} \cup \mathcal{A}^{-1}$.

Note that the Cayley graph is connected, locally finite (that is the degree of every vertex is finite) since $\mathcal{A}$ is finite and Γ acts on it on the left by isomorphisms of graphs. By assigning each edge a metric of length 1, we can define a metric on $C_{\mathcal{A}}(\Gamma)$ which assigns to any pair of vertices (γ, γ') the length of a shortest path from γ to γ'. An expression $\gamma = a_1 a_2 \cdots a_n$ with $a_i \in \mathcal{A} \cup \mathcal{A}^{-1}$ is a *path* in $C_{\mathcal{A}}(\Gamma)$ from 1 to γ and this path is a *geodesic* if $l_{\mathcal{A}}(\gamma) = n$.

Since $\Gamma \hookrightarrow C_{\mathcal{A}}(\Gamma)$, Γ gets a metric induced from the metric on $C_{\mathcal{A}}(\Gamma)$. This metric is exactly the word metric $d_{\mathcal{A}}$ on Γ.

Cayley Graph

Word metrics associated to different finite generating sets $\mathcal{A}$ and $\mathcal{B}$ are *bi-Lipschitz* equivalent,

that is, there exists a constant $\mu > 0$ such that $\frac{1}{\mu} d_{\mathcal{B}}(\gamma, \gamma') \leq d_{\mathcal{A}}(\gamma, \gamma') \leq \mu d_{\mathcal{B}}(\gamma, \gamma')$ for all $\gamma, \gamma' \in \Gamma$.

We can see this by taking $\mu = max\{\mu_1, \mu_2\}$ where $\mu_1 = max\{l_{\mathcal{A}}(b) \mid b \in \mathcal{B}\}, \mu_2 = max\{l_{\mathcal{B}}(a) \mid a \in \mathcal{A}\}$.

Also note that the 1-neighbourhood of Γ in $C\mathcal{A}(\Gamma)$ is the full space. Both Γ and its Cayley graph are the same when viewed from a distance

Coarse Geometry

It is the study of metric spaces from a large-scale point of view so that they look the same from a great distance.
Let us consider what this should mean. One of the ways to make sense of this is to introduce an equivalence relation between metric spaces X and Y which corresponds to them being the same in the large scale.

The right relation to introduce is that of quasi-isometry.

Quasi-isometry

$f : X \to Y$ is a (λ, ϵ)-quasi-isometric embedding (between metric spaces) if:

Definition

$\lambda^{-1} d(x, y) - \epsilon \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \epsilon$ for all $x, y \in X$.

Here $\lambda \geq 1, \epsilon \geq 0$; d, d' are metrics on X and Y respectively.

A (λ, ϵ)-quasi-isometric embedding is a quasi-isometry if there is a $C \geq 0$ such that the $C-$neighbourhood of $f(X)$ is all of Y.

Why quasi-isometry?

This is the right relation for several reasons:

- a compact metric space is coarse equivalent to a point. $(\lambda = 1, \epsilon = M, C = 0)$.
- $\mathbb{Z}$ and $\mathbb{R}$, more generally Γ and $C_{\mathcal{A}}(\Gamma)$ ($\mathcal{A}$ finite) are coarse equivalent. $\mathbb{Z}^n \simeq \mathbb{R}^n$ $\mathbb{Z}^n$ and $\mathbb{Z}^m, n \neq m$ are not.
- For the universal cover of a compact (length) space X, any two metrics that are pullbacks of metrics on X are quasi-isometric.
- $\tilde{X}$ and $\pi_1(X)$, when X is a compact manifold.

Take $\tilde{X} = \mathbb{R}, X = \mathbb{S}^1$ with the geodesic metric and the restricted metric of $\mathbb{R}^2$, then the identity function of $\tilde{X}$ is a quasi-isometry.

Invariants

Quasi-isometric (or large-scale) geometry turns out to be far more rich and powerful than appears at first sight. In fact the most essential elements of an infinite group are quasi-isometry invariants. which means: If the finitely generated group Γ has the property, then every finitely generated group quasi-isometric to Γ also has this property:

- Being finitely presented;
- being virtually infinite cyclic, virtually abelian, virtually nilpotent, virtually free; and
- being hyperbolic!.

Hyperbolic groups

Groups which are coarsely negatively curved.

The notion of hyperbolicity of groups was introduced by Mikhail Gromov.
Let (X, d) be a metric space.
For $\delta \geq 0$, a geodesic triangle is δ-*slim* if each of its sides is contained in the δ-neighbourhood of the union of the other two sides.
A *quasi-geodesic* in X is a quasi-isometric embedding of an interval $I \subset \mathbb{R}$ to X.

$\delta-$hyperbolic space

Definition

A geodesic space X is δ-*hyperbolic* if every geodesic triangle in X is δ-*slim*. The metric space X is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. The exact value of δ is not often relevant for most purposes.

Hyperbolicily is preserved under quasi-isometry between geodesic spaces. See [?].

Hyperbolic Groups

Definition

A finitely generated group is said to be *hyperbolic* if its Cayley graph with respect to some and hence any finite generating set is hyperbolic.

Examples: Finite groups. $\mathbb{Z}$, F_2, F_n.
$\mathrm{SL}(2,\mathbb{Z}), \pi_1(\Sigma_g) \subset \mathrm{PSL}(2,\mathbb{R}), g \geq 2$.
$\mathbb{Z}^2$ is not hyperbolic and hence so is $\mathbb{Z}^n, n \geq 2$

Properties

Hyperbolicity is a quasi isometry invariant.

Not closed under subgroups or taking quotients.

For example, the free group F_2 on 2 generators has its commutator a non-finitely generated subgroup and $\mathbb{Z}^2$ as its abelianization, which are not hyperbolic.

Properties

A hyperbolic group is finitely presented.

The free product $\Gamma_1 * \Gamma_2$ is hyperbolic if Γ_1, Γ_2 are so.
Examples: $\mathbb{Z}_2 * \mathbb{Z}_3 = \mathrm{PSL}(2, \mathbb{Z})$ generated by
$S = z \mapsto -1/z, T = z \mapsto z + 1, S^2 = 1, (ST)^3 = 1$

If Γ is infinite hyperbolic then it has an element of infinite order.

Hence if Γ is a torsion group then it has to be finite.
Also for any element $\gamma \in \Gamma$ of infinite order, the cyclic subgroup $\langle \gamma \rangle$ generated by γ has finite index in its centralizer $C(\gamma)$. Hence an infinite hyperbolic group connot contain $\mathbb{Z}^2$ as a subgroup.

Thank You!