Separation Axioms & Metrizability

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- Metric space satisfies automatically all of the separation

Motivation

Let $A, B \subset X$ be disjoint closed sets. Then there is a continuous function $f : X \to [0, 1]$ with $f_{|A} \equiv 0$, $f_{|B} \equiv 1$, also there are disjoint open sets U, V such that $A \subset U, B \subset V$.

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- Recall: – Convergence of a sequence in a topological space: $x_n \to x$ in X if every neighbourhood U of x contains a tail of (x_n)

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- T_0 property distinguishes points but does not separate points from one another. Not a very useful property.

T_0 spaces

- We can always pass from a non-T_0 space to a T_0-space by passing to a quotient, called Kolmogorov Quotient.
- Let (X, τ) be a topological space.
  Define a relation on X : $x \sim y$ if and only if "for every open set $U \in \tau$, $x \in U \iff y \in U$ "

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- $\sim$ is an equivalence relation on X. (CHECK)
- Let $KQ(X) := \{[x]/x \in X\}$, the set of distinct equivalence classes.

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- A set $U \subset KQ(X)$ is open in $KQ(X)$ iff $\pi^{-1}(U)$ is open in X
  where $\pi^{-1}(V) = \cup_{[x]\in V}[x]$.
  CHECK:
  1. IT GIVES A TOPOLOGY ON KQ(X)
  2. π is an open map ($\pi(U)$ is open for every $U \in \tau$)
  3. $\pi^{-1}(\pi(U)) = U$
- This space is called the Kolmogorov Quotient of X.

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  (otherwise, $[y] \in \pi(U) \implies y \in \pi^{-1}(\pi(U)) = U$, a contradiction)
- $\pi(U)$ is the required open set which distinguishes $[x]$ and $[y]$.

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- If X itself is T_0 then $KQ(X)$ is homeomorphic to X.

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3. $\mathcal{L}^2(\mathbb{R}); = \{f : \mathbb{R} \to \mathbb{R}/f \ \ is \ measurable, \ 2 - integrable\}$ - not T_0 in the semi-norm topology as any two functions that agree on a set of measure zero cannot be distinguished here. $\mathcal{S} = \{B(f, \epsilon) : \epsilon > 0, f \in \mathcal{L}^2(\mathbb{R})\}$ is sub-basis for the topology.

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4. its Kolmogorov Quotient is $L^2(\mathbb{R})$ the Hilbert space we love.
5. If V is a semi-normed space with the topology induced by

1. X is $T_0 \iff$ its Kolmogorov space is T_0
2. X is $T_1 \iff$ Every singleton is closed.
3. X is $T_2 \iff \Delta$ is closed in XxX .
4. X is $T_3 \iff$ for every $x \in X$ and a nbd U of x, there is an open set V such that $x \in V \subset \bar{V} \subset U$.
5. X is $T_4 \iff$ for every closed set A and a nbd U of $A, \exists V$ open such that $A \subset V \subset \bar{V} \subset U$.
6. X is $T_{3\frac{1}{2}} \iff \forall x \in X$ and a nbd U of $x, \exists f \in C(X, [0,1])$ such that $f(x) = 1, f_{|(X-U)} \equiv 0$.

Thank You!